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Algebraic Bethe ansatz for the $gl(1|2)$ generalized model: II. the three gradings

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Abstract

The algebraic Bethe ansatz can be performed rather abstractly for whole classes of models sharing the same R -matrix, the only prerequisite being the existence of an appropriate pseudo vacuum state. Here we perform the algebraic Bethe ansatz for all models with 9×9 , rational, $gl(1|2)$ invariant R -matrix and all three possibilities of choosing the grading. Our Bethe ansatz solution applies, for instance, to the supersymmetric t - J model, the supersymmetric U model and a number of interesting impurity models. It may be extended to obtain the quantum transfer matrix spectrum for this class of models. The properties of a specific model enter the Bethe ansatz solution (i.e. the expression for the transfer matrix eigenvalue and the Bethe ansatz equations) through the three pseudo vacuum eigenvalues of the diagonal elements of the monodromy matrix which in this context are called the parameters of the model.

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1. Introduction

This work resumes previous work of one of the authors [1] where the algebraic Bethe ansatz for the $gl(1|2)$ generalized model was constructed for the grading $(+, -, -)$. In this paper, we address the two remaining cases $(-, +, -)$ and $(-, -, +)$ which turned out to be technically more involved, since the grading enters the auxiliary second level Bethe ansatz in a non-trivial way (see appendix B).

Performing an algebraic Bethe ansatz calculation means to diagonalize the transfer matrix of a certain two-dimensional classical vertex model by purely algebraic means or (in a physicists language) by using only commutation relations between operators. If the transfer matrix has a Hamiltonian limit this is equivalent to diagonalizing the Hamiltonian along with its conserved currents. The Hamiltonian and the conserved currents are then usually generated by expanding the logarithm of the transfer matrix in the spectral parameter.

The algebraic Bethe ansatz can be performed on a considerable level of abstraction and seems to depend only on the structure of the R -matrix of a given model and on the existence of a so-called pseudo vacuum or highest vector [2] on which the monodromy matrix acts as an (upper) triangular matrix. This idea was first of all worked out for the models falling into the same class as the XXZ spin chain [3] and turned out to be useful in calculating the norm [4] and certain matrix elements [5] of Bethe ansatz states. In [3, 4], V E Korepin introduced the notion of a ‘generalized model’ whose ‘representation’ is given by the action of the diagonal elements of the monodromy matrix on the pseudo vacuum. He assumed the vacuum eigenvalues, say $a_1(\lambda)$ and $a_2(\lambda)$, to be arbitrary and called them (functional) parameters of the model. Later [6, 7] V O Tarasov refined and basically confirmed Korepin’s concept.

The algebraic Bethe ansatz for the generalized model associated with the R -matrix of the XXZ model is of the same structure as for the fundamental XXZ spin chain. Such simple relation holds no longer for models of ‘higher rank’ which require a nested Bethe ansatz. The simplest models which allow for a nested algebraic Bethe ansatz are the models with $\mathfrak{gl}(n)$ invariant R -matrix [8, 9]. Considering the fundamental representations of these models, one observes that not only the monodromy matrix elements below the diagonal annihilate the pseudo vacuum, but additional zeros appear above the diagonal [10]. This fact simplifies the algebraic Bethe ansatz for the fundamental representation as compared to the more general case, where the action of all the elements of the monodromy matrix above the diagonal is non-trivial. For the solution of this more general case a new concept, the vacuum subspace, was introduced by Kulish and Reshetikhin [11]. This new concept made it possible to perform the algebraic Bethe ansatz for the models with $\mathfrak{gl}(n)$ invariant R -matrix on the same level of generality as in the $\mathfrak{gl}(2)$ case (corresponding to the R -matrix of the XXX spin chain of spin $\frac{1}{2}$). The resulting eigenvalue of the transfer matrix and the Bethe ansatz equations depend on n functional parameters $a_1(\lambda), \dots, a_n(\lambda)$, which, together with the triangular action of the monodromy matrix on the pseudo vacuum, define the $\mathfrak{gl}(n)$ generalized model [12]. Considering the parameters $a_1(\lambda), a_2(\lambda), a_3(\lambda)$ as free Reshetikhin derived the norm formula for the $\mathfrak{gl}(3)$ case [12].

To our knowledge there was no more activity in the direction of constructing algebraic Bethe ansatz solutions of generalized models beyond the above mentioned work of Kulish and Reshetikhin. This may partially be related to the general difficulties in generalizing the algebraic Bethe ansatz beyond $\mathfrak{gl}(n)$ (see e.g. [13]). For the models with $\mathfrak{gl}(1|2)$ invariant R -matrix (e.g. [14–22]¹), which (together with its anisotropic generalizations) are of particular interest for application in solid state physics, many algebraic Bethe ansatz solutions were constructed [20, 21, 23–29] which can all (except for [26]) be interpreted as certain realizations of the solution obtained in [1] and are extended here to the two gradings $(-, +, -)$ and $(-, -, +)$ not treated in [1].

Future applications of our work may be the quantum transfer matrix approach to the thermodynamics of the models with $\mathfrak{gl}(1|2)$ invariant R -matrix [30, 31], the calculation of norms and matrix elements and, possibly (see [1]), the algebraic Bethe ansatz for the Hubbard model.

Our paper is organized as follows. Section 2 contains the basic definitions relating to the $\mathfrak{gl}(1|2)$ generalized model with some material shifted to appendix A. In section 3, we discuss how the grading and the Yang–Baxter algebra change under permutations of the basis vectors in auxiliary space. We shall see how the grading may change with the change of the highest vector. Section 4 contains our main result, the formulae for the algebraic Bethe ansatz solution of the generalized model for the three possible gradings. For the

¹ For a more thorough discussion see the introduction of [1].

grading $(+, -, -)$ a derivation was presented in [1]. The proof for the remaining two cases is sketched in appendix B. Section 5 contains examples of how to apply the formulae of section 4. We mainly reconsider the well-known example of the fundamental supersymmetric t - J model [24, 25] and clarify the connection between the possible choices of pseudo vacua and possible gradings. We also obtained a simple proof of the equivalence of the different Bethe ansatz solutions of the supersymmetric t - J model (which exceeds the one in [24], since we also show that the eigenvalues are identical). This proof [32] will be published separately. Section 6 contains our conclusion and a discussion of future perspectives.

2. The $\mathfrak{gl}(1|2)$ generalized model

We begin by specifying the class of models we are going to consider. As was explained in the introduction, this class is determined by its R -matrix and by the existence of a pseudo vacuum. Here the R -matrix has matrix elements of the form

$$R_{\beta\delta}^{\alpha\gamma}(\lambda) = a(\lambda)(-1)^{p(\alpha)p(\gamma)}\delta_{\beta}^{\alpha}\delta_{\delta}^{\gamma} + b(\lambda)\delta_{\delta}^{\alpha}\delta_{\beta}^{\gamma} \quad (1)$$

$\alpha, \beta, \gamma, \delta = 1, 2, 3$. This R -matrix is based on graded permutations [15]. It is contained in the early list of Kulish and Sklyanin [8]. The algebraic Bethe ansatz for the fundamental model was constructed by Kulish in 1985 [23] along with the general $\mathfrak{gl}(m|n)$ case.

The R -matrix (1) is characterized by two rational, complex valued functions

$$a(\lambda) = \frac{\lambda}{\lambda + ic} \quad b(\lambda) = \frac{ic}{\lambda + ic} \quad (2)$$

depending on a spectral parameter $\lambda \in \mathbb{C}$ and a coupling $c \in \mathbb{C}$. It further depends on the grading, $p : \{1, 2, 3\} \rightarrow \mathbb{Z}_2$. We shall consider three cases²:

$$\begin{aligned} \text{(i)} \quad & p(1) = 0 \quad p(2) = p(3) = 1 \\ \text{(ii)} \quad & p(2) = 0 \quad p(3) = p(1) = 1 \\ \text{(iii)} \quad & p(3) = 0 \quad p(1) = p(2) = 1. \end{aligned} \quad (3)$$

In order to refer to these different cases we introduce a vector index g which is $g = (g_1, g_2, g_3) = (+, -, -)$ in the first case, $g = (-, +, -)$ in the second case, and $g = (-, -, +)$ in the third case. We shall say ‘the grading is g ’, and we shall write R_g instead of R .

The matrix $R_g(\lambda)$ solves the Yang–Baxter equation and obviously satisfies the compatibility condition [8]

$$R_{g\beta\delta}^{\alpha\gamma}(\lambda) = (-1)^{p(\alpha)+p(\beta)+p(\gamma)+p(\delta)} R_g^{\alpha\gamma}(\lambda). \quad (4)$$

In order to introduce the notion of the graded Yang–Baxter algebra we shall further need the matrix $\check{R}_g(\lambda)$ defined by switching the row indices of $R_g(\lambda)$,

$$\check{R}_{g\beta\delta}^{\alpha\gamma}(\lambda) = R_g^{\gamma\alpha}(\lambda). \quad (5)$$

The graded Yang–Baxter algebra with R -matrix $R(\lambda)$ is the graded, associative algebra (with unity) generated by the elements $T_{\beta}^{\alpha}(\lambda)$, $\alpha, \beta = 1, 2, 3$, of the so-called monodromy matrix modulo the relations

$$\check{R}(\lambda - \mu)(T(\lambda) \otimes_g T(\mu)) = (T(\mu) \otimes_g T(\lambda))\check{R}(\lambda - \mu). \quad (6)$$

We shall assume that the elements of the monodromy matrix are of definite parity, $\pi(T_{\beta}^{\alpha}(\lambda)) = p(\alpha) + p(\beta)$. The symbol \otimes_g denotes the super tensor product associated with the grading g . For a definition see appendix A.

² We comment on the remaining possibilities of choosing the grading below in section 3.

We now define the set of all models related to the R -matrix (1) and solvable by algebraic Bethe ansatz as ‘the $\mathfrak{gl}(1|2)$ generalized model’: by definition the $\mathfrak{gl}(1|2)$ generalized model is the set of all (linear) representations of the graded Yang–Baxter algebra (6) having a highest vector (or pseudo vacuum) Ω . The highest vector Ω is a vector on which the monodromy matrix $\mathcal{T}(\lambda)$ acts as an upper triangular matrix and which is an eigenvector of its diagonal elements:

$$\begin{aligned} \mathcal{T}_1^1(\lambda)\Omega &= a_1(\lambda)\Omega & \mathcal{T}_2^2(\lambda)\Omega &= a_2(\lambda)\Omega & \mathcal{T}_3^3(\lambda)\Omega &= a_3(\lambda)\Omega \\ \mathcal{T}_\beta^\alpha(\lambda)\Omega &= 0 & \text{for } \alpha > \beta. \end{aligned} \quad (7)$$

The eigenvalues $a_j(\lambda)$, $j = 1, 2, 3$ of the diagonal elements of $\mathcal{T}(\lambda)$, are called the parameters of the generalized model. These parameters characterize the representation in a similar manner as the highest weight in a highest weight representation of a Lie algebra.

Let us denote the representation space of a given representation of the generalized model by \mathcal{H} . It is clear from the quadratic commutation relations contained in the graded Yang–Baxter algebra (6) and from (7) that we may assume \mathcal{H} to be spanned by all vectors of the form

$$\Phi(\lambda_1, \dots, \lambda_N) = \mathcal{T}_{\beta_1}^{\alpha_1}(\lambda_1) \dots \mathcal{T}_{\beta_N}^{\alpha_N}(\lambda_N)\Omega \quad (8)$$

where $\alpha_k < \beta_k$, $k = 1, \dots, N$. This assumption is at least sensible for a finite-dimensional representation space \mathcal{H} .

The super trace of the monodromy matrix

$$t(\lambda) = (-1)^{p(\alpha)} \mathcal{T}_\alpha^\alpha(\lambda) = \text{str}_g(\mathcal{T}(\lambda)) \quad (9)$$

is called the transfer matrix of the generalized model. Since $\check{R}(\lambda)$ is invertible for generic values of $\lambda \in \mathbb{C}$, we conclude from (4) and (6) that the transfer matrix satisfies

$$[t(\lambda), t(\mu)] = 0 \quad (10)$$

for all generic $\lambda, \mu \in \mathbb{C}$. It follows that $t(\lambda)$ and $t(\mu)$ have a common system of eigenfunctions, which means that the eigenvectors of $t(\lambda)$ are independent of the spectral parameter λ .

The task to be performed below of the algebraic Bethe ansatz for the generalized model is to diagonalize $t(\lambda)$, i.e., to solve the eigenvalue problem

$$t(\lambda)\Phi = \Lambda(\lambda)\Phi. \quad (11)$$

It is a remarkable fact that this task can be accomplished by solely using the graded Yang–Baxter algebra (6) and properties (7) of the highest vector Ω . In particular, it is *not* necessary to require that $\mathcal{T}_3^2(\lambda)\Omega = 0$ as in case of the fundamental graded representation, which corresponds to the supersymmetric t – J model.

3. Variation of the grading

Before presenting our results for the algebraic Bethe ansatz, we would like to explain why we may restrict ourselves to upper triangular action in our definition (7) of the highest vector Ω , and why we consider only the three gradings shown in equation (3). For the former purpose, we first of all introduce the natural action of the symmetric group \mathfrak{S}^3 on row vectors $\mathbf{x} = (x_1, x_2, x_3)$, setting

$$\mathbf{x}\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \quad (12)$$

for all $\sigma \in \mathfrak{S}^3$. This defines a faithful representation of \mathfrak{S}^3 which is orthogonal with respect to the usual Euclidian scalar product $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$.

$$\begin{array}{ccccc}
 \begin{pmatrix} a_1 & * & * \\ 0 & a_2 & * \\ 0 & 0 & a_3 \end{pmatrix} \Omega & \xrightarrow{\Pi_{12}} & \begin{pmatrix} a_2 & 0 & * \\ * & a_1 & * \\ 0 & 0 & a_3 \end{pmatrix} \Omega & \xrightarrow{\Pi_{23}} & \begin{pmatrix} a_2 & * & 0 \\ 0 & a_3 & 0 \\ * & * & a_1 \end{pmatrix} \Omega \\
 & & & & \downarrow \Pi_{12} \\
 & & & & \begin{pmatrix} a_3 & 0 & 0 \\ * & a_2 & 0 \\ * & * & a_1 \end{pmatrix} \Omega \\
 \uparrow \Pi_{23} & & \xleftarrow{\Pi_{12}} & \begin{pmatrix} a_3 & 0 & 0 \\ * & a_1 & * \\ * & 0 & a_2 \end{pmatrix} \Omega & \xleftarrow{\Pi_{23}} & \begin{pmatrix} a_3 & 0 & 0 \\ * & a_2 & 0 \\ * & * & a_1 \end{pmatrix} \Omega
 \end{array}$$

Figure 1. Change of the monodromy matrix action on the highest vector Ω under permutations of the basis vectors in auxiliary space.

Denoting the transposed of σ by σ^t we obtain the transformation properties of the R -matrix under permutations directly from its definition (1),

$$(\sigma^t \otimes \sigma^t)R_g(\lambda)(\sigma \otimes \sigma) = R_{g\sigma}(\lambda). \tag{13}$$

Similarly, the graded Yang–Baxter algebra (6) is easily seen to transform as

$$\check{R}_{g\sigma}(\lambda - \mu)(\sigma^t T(\lambda)\sigma \otimes_{g\sigma} \sigma^t T(\mu)\sigma) = (\sigma^t T(\mu)\sigma \otimes_{g\sigma} \sigma^t T(\lambda)\sigma)\check{R}_{g\sigma}(\lambda - \mu) \tag{14}$$

while it follows from (9) that

$$\text{str}_g(T(\lambda)) = \text{str}_{g\sigma}(\sigma^t T(\lambda)\sigma) \tag{15}$$

which expresses the invariance of the transfer matrix with respect to permutations.

In figure 1, we show the action of the transformed monodromy matrix $\sigma^t T(\lambda)\sigma$ on a highest vector Ω , when σ runs successively through all permutations in \mathfrak{S}^3 generated by the transpositions of nearest neighbours Π_{12} and Π_{23} . We see that if, for a given grading g , a monodromy matrix $T(\lambda)$ realizes one of the six patterns in figure 1 by acting on some vector Ω then there is a permutation $\sigma \in \mathfrak{S}^3$ such that $\sigma^t T(\lambda)\sigma$ acts as an upper triangular matrix on Ω . The corresponding grading changes from g to $g\sigma$.

Let us consider an example. Take $g = (-, -, +)$ and $T(\lambda)$ and Ω such that

$$T(\lambda)\Omega = \begin{pmatrix} a_1 & 0 & 0 \\ * & a_2 & * \\ * & 0 & a_3 \end{pmatrix} \Omega. \tag{16}$$

This is the pattern in the middle of the second row in figure 1. Thus,

$$\Pi_{23}\Pi_{12}T(\lambda)\Pi_{12}\Pi_{23}\Omega = \begin{pmatrix} a_2 & * & * \\ 0 & a_3 & * \\ 0 & 0 & a_1 \end{pmatrix} \Omega \tag{17}$$

and the grading changes to $(-, -, +)\Pi_{12}\Pi_{23} = (-, +, -)$.

Combining the six patterns in figure 1 with the three gradings $g = (+, -, -)$, $(-, +, -)$, $(-, -, +)$ we obtain 18 cases which by application of permutations all reduce back to three, e.g. upper triangular action with three different gradings. The Bethe ansatz solutions of the transfer matrix eigenvalue problem (11) for these three cases will be presented in the following section.

Note that it may happen that there are several vectors $\Omega_1, \Omega_2, \dots$ which for given monodromy matrix and grading generate several of the patterns in figure 1. Then there are several equivalent but differently looking Bethe ansatz solutions of the transfer matrix

eigenvalue problem. This is, for instance, the case for the supersymmetric t - J model as was observed in [24]. We will come back to this phenomenon in our example section (see section 5).

How about the other possible gradings? There are eight cases altogether. Three cases are listed in (3). The case $g = (+, +, +)$ was treated by Kulish and Reshetikhin [11]. The four remaining cases are related to our work or the work of Kulish and Reshetikhin by a switch of sign of g (e.g. $(+, -, -) \rightarrow (-, +, +)$). It is easy to see that this switch modifies the Yang–Baxter algebra (6) only trivially: we introduce a diagonal matrix $G = \text{diag}(g_1, g_2, g_3)$ and the 3×3 unit matrix I_3 . Then a similarity transformation with $G \otimes I_3$ transforms the graded Yang–Baxter algebra (6) with R -matrix R_g into

$$\check{R}_{-g}(\lambda - \mu)(T(-\lambda) \otimes_{-g} T(-\mu)) = (T(-\mu) \otimes_{-g} T(-\lambda))\check{R}_{-g}(\lambda - \mu). \tag{18}$$

Note that the expression for the parity of the monodromy matrix elements, $\pi(T_\beta^\alpha(\lambda)) = p(\alpha) + p(\beta)$, is invariant under a change of the sign of g , since it corresponds to replacing $p(\alpha)$ by $p(\alpha) + 1$. Thus, every representation with parameters $a_j(\lambda)$ of the graded Yang–Baxter algebra (6) with grading g and R -matrix $R_g(\lambda)$ is equivalent to a representation with parameters $a_j(-\lambda)$ of (6) with grading $-g$ and R -matrix $R_{-g}(\lambda)$. Consequently the Bethe ansatz solutions of the generalized model for the remaining gradings are obtained from the solutions in the following section by switching the sign in the argument of $a(\lambda)$ and the overall signs of the transfer matrix eigenvalues (or by performing similar manipulations in the corresponding equations in [11]).

4. The algebraic Bethe ansatz solution

In this section, we list the transfer matrix eigenvalues and the corresponding eigenvectors. For $g = (+, -, -)$ they were obtained in [1]. The derivations for the two remaining cases $g = (-, +, -)$ and $g = (-, -, +)$ are presented in appendix B.

Recall that the functions a_1, a_2 and a_3 are the (functional) parameters of the model and that $a(\lambda) = \lambda/(\lambda + ic)$. The different transfer matrix eigenvalues $\Lambda_g(\lambda)$ (equations (19), (21) and (23)) are distinguished for a given grading by specifying two sets of Bethe roots $\{\lambda_j\}_{j=1}^N$ and $\{\mu_k\}_{k=1}^M$ which have to be calculated from two coupled sets of Bethe ansatz equations (see (20), (22), (24)).

$$\begin{aligned} \Lambda_{(+--)}(\lambda) = & a_1(\lambda) \prod_{j=1}^N \frac{1}{a(\lambda_j - \lambda)} \\ & - a_2(\lambda) \prod_{j=1}^N \frac{1}{a(\lambda_j - \lambda)} \prod_{k=1}^M \frac{1}{a(\lambda - \mu_k)} - a_3(\lambda) \prod_{k=1}^M \frac{1}{a(\mu_k - \lambda)} \end{aligned} \tag{19}$$

$$\frac{a_1(\lambda_j)}{a_2(\lambda_j)} = \prod_{k=1}^M \frac{1}{a(\lambda_j - \mu_k)} \quad j = 1, \dots, N \tag{20a}$$

$$\frac{a_3(\mu_k)}{a_2(\mu_k)} = \prod_{\substack{l=1 \\ l \neq k}}^M \frac{a(\mu_l - \mu_k)}{a(\mu_k - \mu_l)} \prod_{j=1}^N \frac{1}{a(\lambda_j - \mu_k)} \quad k = 1, \dots, M. \tag{20b}$$

$$\Lambda_{(-+-)}(\lambda) = -a_1(\lambda) \prod_{j=1}^N \frac{1}{a(\lambda - \lambda_j)} + a_2(\lambda) \prod_{j=1}^N \frac{1}{a(\lambda - \lambda_j)} \prod_{k=1}^M \frac{1}{a(\mu_k - \lambda)} - a_3(\lambda) \prod_{k=1}^M \frac{1}{a(\mu_k - \lambda)} \quad (21)$$

$$\frac{a_1(\lambda_j)}{a_2(\lambda_j)} = \prod_{k=1}^M \frac{1}{a(\mu_k - \lambda_j)} \quad j = 1, \dots, N \quad (22a)$$

$$\frac{a_3(\mu_k)}{a_2(\mu_k)} = \prod_{j=1}^N \frac{1}{a(\mu_k - \lambda_j)} \quad k = 1, \dots, M. \quad (22b)$$

$$\Lambda_{(---)}(\lambda) = -a_1(\lambda) \prod_{j=1}^N \frac{1}{a(\lambda - \lambda_j)} - a_2(\lambda) \prod_{j=1}^N \frac{1}{a(\lambda_j - \lambda)} \prod_{k=1}^M \frac{1}{a(\lambda - \mu_k)} + a_3(\lambda) \prod_{k=1}^M \frac{1}{a(\lambda - \mu_k)} \quad (23)$$

$$\frac{a_1(\lambda_j)}{a_2(\lambda_j)} = \prod_{\substack{l=1 \\ l \neq j}}^N \frac{a(\lambda_j - \lambda_l)}{a(\lambda_l - \lambda_j)} \prod_{k=1}^M \frac{1}{a(\lambda_j - \mu_k)} \quad j = 1, \dots, N \quad (24a)$$

$$\frac{a_3(\mu_k)}{a_2(\mu_k)} = \prod_{j=1}^N \frac{1}{a(\lambda_j - \mu_k)} \quad k = 1, \dots, M. \quad (24b)$$

These three sets of expressions for the eigenvalues and Bethe ansatz equations depend on the grading $g = (g_1, g_2, g_3)$ in a characteristic way which allows us to write them all in one (for a similarly compact expression for the (q -deformed) fundamental model see [33]):

$$\Lambda_g(\lambda) = g_1 a_1(\lambda) \prod_{j=1}^N \frac{1}{a(g_1(\lambda_j - \lambda))} + g_2 a_2(\lambda) \prod_{j=1}^N \frac{1}{a(g_2(\lambda - \lambda_j))} \prod_{k=1}^M \frac{1}{a(g_2(\mu_k - \lambda))} + g_3 a_3(\lambda) \prod_{k=1}^M \frac{1}{a(g_3(\lambda - \mu_k))} \quad (25)$$

$$\frac{a_1(\lambda_j)}{a_2(\lambda_j)} = \prod_{\substack{l=1 \\ l \neq j}}^N \frac{a(g_1(\lambda_l - \lambda_j))}{a(g_2(\lambda_j - \lambda_l))} \prod_{k=1}^M \frac{1}{a(g_2(\mu_k - \lambda_j))} \quad j = 1, \dots, N \quad (26a)$$

$$\frac{a_3(\mu_k)}{a_2(\mu_k)} = \prod_{\substack{l=1 \\ l \neq k}}^M \frac{a(g_3(\mu_k - \mu_l))}{a(g_2(\mu_l - \mu_k))} \prod_{j=1}^N \frac{1}{a(g_2(\mu_k - \lambda_j))} \quad k = 1, \dots, M. \quad (26b)$$

Our notation means, for instance, that by specifying $g_1 = -1, g_2 = 1, g_3 = -1$ equations (25) and (26) turn into (21) and (22) corresponding to the grading $g = (-, +, -)$.

Describing the corresponding eigenvectors requires more effort, since we will have to introduce several notions related to the ‘second Bethe ansatz’ in the nested Bethe ansatz

calculation that led to the above expressions for the eigenvalues. The eigenvectors are obtained by acting with certain linear combinations of products of monodromy matrix elements on the highest vector Ω . They are of the form

$$\Phi_g(\lambda_1, \dots, \lambda_N; \mu_1, \dots, \mu_M) = \sum_{i_1, \dots, i_N=1}^2 B_{i_1}^g(\lambda_1) \dots B_{i_N}^g(\lambda_N) \varphi_g^{i_1 \dots i_N}(\mu_1, \dots, \mu_M) \Omega. \quad (27)$$

Here the $B_j^g, j = 1, 2$, may be thought of as components of row vectors

$$B^{(+-)}(\lambda) = (T_2^1(\lambda), T_3^1(\lambda)) \quad (28a)$$

$$B^{(-+-)}(\lambda) = (T_3^1(\lambda), T_2^1(\lambda)) \quad (28b)$$

$$B^{(--+)}(\lambda) = (T_2^1(\lambda), T_3^1(\lambda)). \quad (28c)$$

φ_g is defined in terms of an auxiliary monodromy matrix which is a product of two 2×2 matrices

$$\tilde{T}(\lambda) = \begin{pmatrix} \tilde{A}(\lambda) & \tilde{B}(\lambda) \\ \tilde{C}(\lambda) & \tilde{D}(\lambda) \end{pmatrix} = \mathcal{D}_g(\lambda) \mathcal{T}_{g'}(\lambda). \quad (29)$$

The factor $\mathcal{D}_g(\lambda)$ basically contains elements of the monodromy matrix $\mathcal{T}(\lambda)$,

$$\mathcal{D}_{(+-)}(\lambda) = \begin{pmatrix} T_2^2(\lambda) & T_3^2(\lambda) \\ T_2^3(\lambda) & T_3^3(\lambda) \end{pmatrix} \quad (30a)$$

$$\mathcal{D}_{(-+-)}(\lambda) = \begin{pmatrix} T_3^3(\lambda) & T_2^3(\lambda)(\sigma^z)^{\otimes N} \\ T_3^2(\lambda)(\sigma^z)^{\otimes N} & T_2^2(\lambda) \end{pmatrix} \quad (30b)$$

$$\mathcal{D}_{(--+)}(\lambda) = \begin{pmatrix} T_2^2(\lambda) & T_3^2(\lambda)(\sigma^z)^{\otimes N} \\ T_2^3(\lambda)(\sigma^z)^{\otimes N} & T_3^3(\lambda) \end{pmatrix}. \quad (30c)$$

The factor $\mathcal{T}_{g'}(\lambda)$ is the monodromy matrix of an auxiliary ‘spin problem’,

$$\mathcal{T}_{g'}(\lambda) = \mathcal{L}_N^{g'}(\lambda_N - \lambda) \dots \mathcal{L}_1^{g'}(\lambda_1 - \lambda) \quad (31)$$

carrying an induced grading g' , which is $(+, +)$ for $g = (+, -, -)$ and $(+, -)$ for the remaining two cases $g = (-, +, -)$ and $g = (-, -, +)$. The corresponding elementary L -matrices are³

$$\mathcal{L}_j^{(++)}(\lambda) = a(\lambda)I_2 + b(\lambda) \begin{pmatrix} e_{j1}^1 & e_{j2}^1 \\ e_{j1}^2 & e_{j2}^2 \end{pmatrix}. \quad (32)$$

and

$$\mathcal{L}_j^{(+)}(\lambda) = a(\lambda)I_2 + b(\lambda) \begin{pmatrix} e_{j1}^1 & e_{j2}^1 \\ e_{j1}^2 & -e_{j2}^2 \end{pmatrix}. \quad (33)$$

The $e_{j\alpha}^\beta$ are the canonical basis elements of $(\text{End}(\mathbb{C}^2))^{\otimes N}$ (viewed as a graded algebra) introduced in appendix A. They depend on the induced grading g' . The L -matrix $\mathcal{L}_j^{(++)}$ is the L -matrix of the XXX spin chain of spin 1/2, whereas $\mathcal{L}_j^{(+-)}$ generates the $\text{gl}(1|1)$ invariant model of free spinless fermions.

³ Note that, as compared to [1], we have changed the definition of the L -matrix $\mathcal{L}_j^{(++)}$ into the equivalent form (32).

Inserting (30)–(33) into (29) we have defined the auxiliary monodromy matrix $\tilde{T}(\lambda)$ for the three different gradings under consideration. The 2^N -column vector φ_g in equation (27) is constructed by acting with matrix elements of $\tilde{T}(\lambda)$ on appropriate auxiliary states. For $g = (+, -, -)$ and $g = (-, -, +)$ we define

$$\varphi_g(\mu_1, \dots, \mu_M) = \tilde{B}(\mu_1) \dots \tilde{B}(\mu_M) \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes N} \tag{34}$$

for $g = (-, +, -)$ an appropriate definition is

$$\varphi_g(\mu_1, \dots, \mu_M) = \tilde{C}(\mu_1) \dots \tilde{C}(\mu_M) \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes N}. \tag{35}$$

A derivation of the eigenvectors and the corresponding eigenvalues for $g = (-, +, -)$ and $g = (-, -, +)$ is presented in appendix B. The proof for $g = (+, -, -)$ can be found in [1].

5. Examples

Numerous examples of systems for which the Bethe ansatz solution of the previous section applies can be constructed starting from the observation [23] that

$$\mathcal{L}_{j\beta}^\alpha(\lambda) = a(\lambda)\delta_\beta^\alpha + b(\lambda)(-1)^{p(\alpha)p(\beta)} E_{j\beta}^\alpha \tag{36}$$

with $a(\lambda), b(\lambda)$ defined in (2) is a representation of the graded Yang–Baxter algebra (6) if $E_{j\beta}^\alpha$ is a representation of $gl(1|2)$ of parity compatible with the grading,

$$[E_{j\beta}^\alpha, E_{k\delta}^\gamma]_\pm = \delta_{jk}(\delta_\delta^\alpha E_{j\beta}^\gamma - (-1)^{(p(\alpha)+p(\beta))(p(\gamma)+p(\delta))} \delta_\beta^\gamma E_{j\delta}^\alpha). \tag{37}$$

Here $[\cdot, \cdot]_\pm$ denotes the super-bracket (see appendix A).

For the construction of models of fermions on one-dimensional lattices (which is our personal concern with the Bethe ansatz presented in the previous chapter) one may generally utilize (36) in a way that involves three logically separate steps:

- (i) Take a representation of $gl(1|2)$ in $\text{End}(\mathbb{C}^n)$ (equation (37) with $j = k$, see [34, 35]).
- (ii) Embed it into $(\text{End}(\mathbb{C}^n))^{\otimes L}$ in such a way that the grading ((37) for $j \neq k$) is respected [36].
- (iii) Introduce Fermi operators [36, 37].

Since we merely want to illustrate our Bethe ansatz solution of the previous sections, we shall take the three steps in one in the examples considered below.

The most elementary example is, of course, the supersymmetric t – J model [16, 24] which is the fundamental model associated with the R -matrix (1). The supersymmetric t – J model is a model of electrons on a lattice. In order to be able to write down the L -matrix and the Hamiltonian in a familiar way we introduce canonically anticommuting creation and annihilation operators $c_{j,a}^\dagger, c_{k,b}$ where the indices $j, k = 1, \dots, L$ refer to the lattice sites, and $a, b = \uparrow, \downarrow$ are spin indices.

Due to the canonical anticommutation relations the elements $(X_j)_\beta^\alpha, \alpha, \beta = 1, 2, 3$, of the matrix

$$X_j = \begin{pmatrix} (1 - n_{j\downarrow})(1 - n_{j\uparrow}) & (1 - n_{j\downarrow})c_{j\uparrow} & c_{j\downarrow}(1 - n_{j\uparrow}) \\ (1 - n_{j\downarrow})c_{j\uparrow}^\dagger & (1 - n_{j\downarrow})n_{j\uparrow} & -c_{j\downarrow}c_{j\uparrow}^\dagger \\ c_{j\downarrow}^\dagger(1 - n_{j\uparrow}) & c_{j\downarrow}^\dagger c_{j\uparrow} & n_{j\downarrow}(1 - n_{j\uparrow}) \end{pmatrix} \tag{38}$$

with $n_{j,\uparrow} = c_{j,\uparrow}^\dagger c_{j,\uparrow}, n_{j,\downarrow} = c_{j,\downarrow}^\dagger c_{j,\downarrow}$ form a complete set of ‘projection operators’ on the space of states locally spanned by the basis vectors $|0\rangle, c_{j\uparrow}^\dagger|0\rangle, c_{j\downarrow}^\dagger|0\rangle$. Double occupancy of lattice

sites is forbidden on this space. Setting $X_{j\alpha}^\beta = (X_j)^\alpha_\beta$, $\alpha, \beta = 1, 2, 3$, we find

$$X_{j\alpha}^\beta X_{j\gamma}^\delta = \delta_\gamma^\beta X_{j\alpha}^\delta \tag{39a}$$

$$X_{j\alpha}^\beta X_{k\gamma}^\delta = (-1)^{(p(\alpha)+p(\beta))(p(\gamma)+p(\delta))} X_{k\gamma}^\delta X_{j\alpha}^\beta \quad \text{for } j \neq k \tag{39b}$$

where $p(1) = 0, p(2) = p(3) = 1$. It follows that the operators $X_{j\alpha}^\beta$ satisfy equation (37). The linear combination $X_{j\alpha}^\alpha = 1 - n_{j\uparrow}n_{j\downarrow}$ projects the local space of lattice electrons onto the space from which the double occupancy is excluded. The corresponding global projection operator is

$$P_0 = \prod_{j=1}^L (1 - n_{j\uparrow}n_{j\downarrow}). \tag{40}$$

It will be needed below.

We conclude with (36), (37) and (39) that the L -matrix

$$\mathcal{L}_j(\lambda) = a(\lambda)I_3 + b(\lambda) \begin{pmatrix} X_{j1}^1 & X_{j2}^1 & X_{j3}^1 \\ X_{j1}^2 & -X_{j2}^2 & -X_{j3}^2 \\ X_{j1}^3 & -X_{j2}^3 & -X_{j3}^3 \end{pmatrix} \tag{41}$$

is a representation of the graded Yang–Baxter algebra (6) with grading $(+, -, -)$. This representation has been termed fundamental graded representation in [36]. The Fock vacuum $|0\rangle$ satisfying $c_{j,a}|0\rangle = 0$ for $j = 1, \dots, L$; $a, b = \uparrow, \downarrow$ is clearly a possible highest vector for $\mathcal{L}_j(\lambda)$,

$$\mathcal{L}_j(\lambda)|0\rangle = \begin{pmatrix} 1 & b(\lambda)X_{j2}^1 & b(\lambda)X_{j3}^1 \\ 0 & a(\lambda) & 0 \\ 0 & 0 & a(\lambda) \end{pmatrix} |0\rangle. \tag{42}$$

It turns out that the matrix $\mathcal{L}_j(\lambda)$ generates the supersymmetric t – J model at a single site. The corresponding monodromy matrix of the L -site model is

$$\mathcal{T}(\lambda) = \mathcal{L}_L(\lambda) \dots \mathcal{L}_1(\lambda). \tag{43}$$

Its action on the Fock vacuum follows from (42) as

$$\mathcal{T}(\lambda)|0\rangle = \begin{pmatrix} 1 & B_1(\lambda) & B_2(\lambda) \\ 0 & a^L(\lambda) & 0 \\ 0 & 0 & a^L(\lambda) \end{pmatrix} |0\rangle. \tag{44}$$

Thus, $\mathcal{T}(\lambda)$ is a representation of the graded Yang–Baxter algebra (6), and $|0\rangle$ is a highest vector satisfying (7). It follows that our general formulae (19) and (20) apply with functional parameters which can be read off from (44):

$$a_1(\lambda) = 1 \quad a_2(\lambda) = a_3(\lambda) = a^L(\lambda). \tag{45}$$

This way we have recovered equations (3.47), (3.48) and (3.50) of [24].

Note that the Hamiltonian of the supersymmetric t – J model is

$$H = -ic\partial_\lambda \ln\{(\text{str}(\mathcal{T}(0)))^{-1} \text{str}(\mathcal{T}(\lambda))\}|_{\lambda=0}. \tag{46}$$

Because it acts on the restricted space of electronic states, where no lattice site is doubly occupied, we may replace it with (see [36, 37])

$$HP_0 = P_0 \left\{ - \sum_{j=1}^L (c_{j,a}^\dagger c_{j+1,a} + c_{j+1,a}^\dagger c_{j,a}) + 2 \sum_{j=1}^L \left(S_j^\alpha S_{j+1}^\alpha - \frac{n_j n_{j+1}}{4} + n_j \right) \right\} P_0 \tag{47}$$

where we inserted the usual definitions $S_j^\alpha = \frac{1}{2} \sum_{a,b=\uparrow,\downarrow} c_{j,a}^\dagger \sigma_{ab}^\alpha c_{j,b}$ of local spin operators in terms of Pauli matrices σ^α and introduced the local particle number operators $n_j = n_{j,\uparrow} + n_{j,\downarrow}$.

Clearly the Fock vacuum is also a highest vector for $\Pi_{23}\mathcal{T}(\lambda)\Pi_{23}$, since under a permutation of the second and third basis vector in auxiliary space $\mathcal{L}_j(\lambda)|0\rangle$ transforms into

$$\Pi_{23}\mathcal{L}_j(\lambda)\Pi_{23}|0\rangle = \begin{pmatrix} 1 & b(\lambda)X_{j3}^1 & b(\lambda)X_{j2}^1 \\ 0 & a(\lambda) & 0 \\ 0 & 0 & a(\lambda) \end{pmatrix} |0\rangle \tag{48}$$

which is of upper triangular form. The monodromy matrices $\mathcal{T}(\lambda)$ and $\Pi_{23}\mathcal{T}(\lambda)\Pi_{23}$ carry the same grading and, as for arbitrary permutations, lead to the same transfer matrix (see equation (15)). Since the grading as well as the parameters a_1, a_2 and a_3 is identical for $\mathcal{T}(\lambda)$ and $\Pi_{23}\mathcal{T}(\lambda)\Pi_{23}$, both monodromy matrices lead to the same form (19) of the transfer matrix eigenvalue and to the same Bethe ansatz equations (20). Note, however, that the Bethe ansatz eigenvectors (27) are different, because $B^{(+--)}(\lambda)$ is changed to $B^{(+--)}(\lambda)\sigma^x$ which is equivalent to a spin flip transformation.

Indeed, applying the spin flip transformation

$$J^{(s)} = \prod_{j=1}^L (1 - (c_{j,\uparrow}^\dagger - c_{j,\downarrow}^\dagger)(c_{j,\uparrow} - c_{j,\downarrow})) \tag{49}$$

to the elementary L -matrix we obtain

$$J^{(s)}\mathcal{L}_j(\lambda)(J^{(s)})^\dagger = \Pi_{23}\mathcal{L}_j(\lambda)\Pi_{23} \tag{50}$$

which implies the invariance of the transfer matrix with respect to spin flips.

It is a well-known fact that there are three alternative sets of Bethe ansatz equations for the supersymmetric t - J model [24, 25]. Let us see how this comes out in our general formalism. We shall consider the monodromy matrix action on the two states

$$|\uparrow\rangle = X_{L2}^1 \dots X_{12}^1 |0\rangle = c_{L,\uparrow}^\dagger \dots c_{1,\uparrow}^\dagger |0\rangle \tag{51a}$$

$$|\downarrow\rangle = X_{L3}^1 \dots X_{13}^1 |0\rangle = c_{L,\downarrow}^\dagger \dots c_{1,\downarrow}^\dagger |0\rangle. \tag{51b}$$

Calculating first of all the action of the L -matrix on these states we obtain

$$\mathcal{L}_j(\lambda)|\uparrow\rangle = \begin{pmatrix} a(\lambda) & 0 & 0 \\ b(\lambda)X_{j1}^2 & a(\lambda) - b(\lambda) & -b(\lambda)X_{j3}^2 \\ 0 & 0 & a(\lambda) \end{pmatrix} |\uparrow\rangle \tag{52}$$

$$\mathcal{L}_j(\lambda)|\downarrow\rangle = \begin{pmatrix} a(\lambda) & 0 & 0 \\ 0 & a(\lambda) & 0 \\ b(\lambda)X_{j1}^3 & -b(\lambda)X_{j2}^3 & a(\lambda) - b(\lambda) \end{pmatrix} |\downarrow\rangle. \tag{53}$$

Comparing the patterns of zeros on the right-hand side of these equations to figure 1 we see that $|\uparrow\rangle$ is a highest vector for $\Pi_{12}\mathcal{T}(\lambda)\Pi_{12}$ and for $\Pi_{23}\Pi_{12}\mathcal{T}(\lambda)\Pi_{12}\Pi_{23}$, while $|\downarrow\rangle$ is a highest vector for $\Pi_{12}\Pi_{23}\mathcal{T}(\lambda)\Pi_{23}\Pi_{12}$ and for $\Pi_{12}\Pi_{23}\Pi_{12}\mathcal{T}(\lambda)\Pi_{12}\Pi_{23}\Pi_{12}$. Together with the possibilities already covered by using $|0\rangle$ as a highest vector we obtain all six cases of figure 1 albeit with different grading. The situation is summarized in table 1.

Taking the data from the table and inserting it into (25)–(27) we obtain the different possible Bethe ansatz solutions of the supersymmetric t - J model. The expressions for the eigenvalues and Bethe ansatz equations are in agreement with the results of [24]. Because of space limitations we do not repeat those results here. In [24] and also in [25], the authors

Table 1. Different Bethe ansatz solutions for supersymmetric t - J model.

$\sigma \in \mathfrak{S}^3$	Ω	g	$a_1(\lambda)$	$a_2(\lambda)$	$a_3(\lambda)$
id	$ 0\rangle$	$(+, -, -)$	1	$a^L(\lambda)$	$a^L(\lambda)$
Π_{12}	$ \uparrow\rangle$	$(-, +, -)$	$\frac{a^L(\lambda)}{a^L(-\lambda)}$	$a^L(\lambda)$	$a^L(\lambda)$
$\Pi_{12}\Pi_{23}$	$ \downarrow\rangle$	$(-, +, -)$	$\frac{a^L(\lambda)}{a^L(-\lambda)}$	$a^L(\lambda)$	$a^L(\lambda)$
$\Pi_{12}\Pi_{23}\Pi_{12}$	$ \downarrow\rangle$	$(-, -, +)$	$\frac{a^L(\lambda)}{a^L(-\lambda)}$	$a^L(\lambda)$	$a^L(\lambda)$
$\Pi_{23}\Pi_{12}$	$ \uparrow\rangle$	$(-, -, +)$	$\frac{a^L(\lambda)}{a^L(-\lambda)}$	$a^L(\lambda)$	$a^L(\lambda)$
Π_{23}	$ 0\rangle$	$(+, -, -)$	1	$a^L(\lambda)$	$a^L(\lambda)$

avoided writing explicit expressions for the L -matrix in terms of Fermi operators. Therefore they could not see the correspondence between the possible choices of pseudo vacua $|0\rangle, |\uparrow\rangle, |\downarrow\rangle$ and the possible gradings. As we can learn from table 1 this correspondence is not unique. For the highest vector $|\uparrow\rangle$ the Bethe ansatz can be realized with grading $(-, +, -)$ or $(-, -, +)$, respectively. A similar statement holds for $|\downarrow\rangle$. By way of contrast, the Bethe ansatz equations and the expression for the transfer matrix eigenvalue are uniquely fixed if we choose $|0\rangle$ as the highest vector. Still, as was observed above, the eigenvectors can be realized in two different ways.

The algebraic Bethe ansatz for the supersymmetric t - J model is rather special as compared to the general case. This is due to the fact that $a_2(\lambda) = a_3(\lambda)$ (see table 1) and that an additional zero appears above the diagonal in the action of the monodromy matrix on the highest vector. Consider, for instance, the case $\Omega = |0\rangle, \sigma = \text{id}$. Then the monodromy matrix action on Ω is given by (44). Making contact with the notation of (30) we see that $\mathcal{D}_{(+--)}(\lambda)|0\rangle = a^L(\lambda)I_2|0\rangle$. Because of this trivial action, we may drop the factor $\mathcal{D}_g(\lambda)$ on the right-hand side of (29) and the eigenvectors (27) are written only in terms of $\mathcal{T}_2^1(\lambda), \mathcal{T}_3^1(\lambda)$ and the auxiliary monodromy matrix $\mathcal{T}_{(++)}(\lambda)$ (see (31) and (32)).

More examples are obtained by inserting other representations of (37) into (36). For the grading $g = (+, -, -)$ we may, for instance, take the four-dimensional representation

$$E = \begin{pmatrix} \text{ch}^2(\alpha) - n_\uparrow - n_\downarrow & \text{ch}(\alpha)c_\uparrow^\dagger - e^{-\alpha}c_\uparrow^\dagger n_\downarrow & \text{ch}(\alpha)c_\downarrow^\dagger - e^{-\alpha}c_\downarrow^\dagger n_\uparrow \\ \text{ch}(\alpha)c_\uparrow - e^{-\alpha}c_\uparrow n_\downarrow & n_\uparrow & c_\downarrow^\dagger c_\uparrow \\ \text{ch}(\alpha)c_\downarrow - e^{-\alpha}c_\downarrow n_\uparrow & c_\uparrow^\dagger c_\downarrow & n_\downarrow \end{pmatrix} \tag{54}$$

which depends on a free parameter $\alpha \in \mathbb{R}$. Note however, that it requires more effort [2] to associate a physically relevant model having a local Hamiltonian with higher dimensional representations. This is a subject that exceeds the scope of this work.

6. Conclusion

We have completed the task, begun in [1], of constructing the algebraic Bethe ansatz for the $\mathfrak{gl}(1|2)$ generalized model. In this work, the gradings $g = (-, +, -)$ and $g = (-, -, +)$ have been treated. As we hope to have convinced the reader in sections 3 and 5, a complete understanding requires to consider the three gradings together.

We hope our work will prove to be useful in future constructions of exact solutions of models with the R -matrix (1). In first place we think of novel impurity models and of possible applications to Yang’s model of electrons interacting via delta function potential and, maybe, to the Hubbard model (see discussion in [1]). Other applications may be the calculation of

quantum transfer matrix eigenvalues (the quantum transfer matrix has a staggered pseudo vacuum on which the monodromy matrix acts ‘without producing additional zeros above the diagonal’) and the calculation of norms of Bethe ansatz eigenstates (see [1]).

Appendix A. Graded algebras

In this appendix, we shall recall the basic concepts of graded vector spaces and graded associative algebras. In the context of the quantum inverse scattering method, these concepts were first utilized by Kulish and Sklyanin [8, 23].

Graded vector spaces are vector spaces equipped with a notion of odd and even, that allows us to treat fermions within the formalism of the quantum inverse scattering method (see [36, 37]). Let us consider a finite-dimensional vector space V , which is the direct sum of two subspaces, $V = V_0 \oplus V_1$, $\dim V_0 = m$, $\dim V_1 = n$. We shall call $v_0 \in V_0$ even and $v_1 \in V_1$ odd. The subspaces V_0 and V_1 are called the homogeneous components of V . The parity π is a function $V_i \rightarrow \mathbb{Z}_2$ defined on the homogeneous components of V ,

$$\pi(v_i) = i \quad i = 0, 1 \quad v_i \in V_i. \tag{A.1}$$

The vector space V endowed with this structure is called a graded vector space or super space.

Let \mathcal{A} be an associative algebra (with unity), which is graded as a vector space. Suppose $X, Y \in \mathcal{A}$ are homogeneous. If the product XY is homogeneous with parity

$$\pi(XY) = \pi(X) + \pi(Y) \tag{A.2}$$

then \mathcal{A} is called a graded associative algebra [8].

For any two homogeneous elements $X, Y \in \mathcal{A}$ let us define the super-bracket

$$[X, Y]_{\pm} = XY - (-1)^{\pi(X)\pi(Y)} YX \tag{A.3}$$

and let us extend this definition linearly in both of its arguments to all elements of \mathcal{A} .

Let $p : \{1, \dots, n\} \rightarrow \mathbb{Z}_2$. The set of all $n \times n$ matrices A, B, C, \dots with entries in \mathcal{A} , such that $\pi(A_{\beta}^{\alpha}) = \pi(B_{\beta}^{\alpha}) = \pi(C_{\beta}^{\alpha}) = \dots = p(\alpha) + p(\beta)$ is an associative algebra, say $\text{Mat}(\mathcal{A}, n)$, since $\pi(A_{\beta}^{\alpha} B_{\gamma}^{\beta}) = p(\alpha) + p(\gamma)$. For $A, B \in \text{Mat}(\mathcal{A}, n)$ we define the graded tensor product (or super tensor product)

$$(A \otimes_g B)_{\beta\delta}^{\alpha\gamma} = (-1)^{(p(\alpha)+p(\beta))p(\gamma)} A_{\beta}^{\alpha} B_{\delta}^{\gamma}. \tag{A.4}$$

The graded tensor product is associative. For matrices $A, B, C, D \in \text{Mat}(\mathcal{A}, n)$ with $[B_{\beta}^{\alpha}, C_{\delta}^{\gamma}]_{\pm} = 0$ we find

$$(A \otimes_g B)(C \otimes_g D) = AC \otimes_g BD. \tag{A.5}$$

Our chief example of a graded associative algebra is the algebra $\mathcal{A} = (\text{End}(V))^{\otimes L}$, where V is a graded vector space as introduced above. \mathcal{A} acquires the structure of a graded algebra in the following way: we fix a basis $\{e_1, \dots, e_{m+n}\}$ of definite parity and define $p : \{1, \dots, m+n\} \rightarrow \mathbb{Z}_2$ by setting $p(\alpha) = \pi(e_{\alpha})$. Then the set $\{e_{\alpha}^{\beta} \in \text{End}(V) | \alpha, \beta = 1, \dots, m+n\}$ with $e_{\alpha}^{\beta} e_{\gamma} = \delta_{\gamma}^{\beta} e_{\alpha}$ is a basis of $\text{End}(V)$, and the tensor products $e_{\alpha_1}^{\beta_1} \otimes \dots \otimes e_{\alpha_L}^{\beta_L}$ span the vector space $\mathcal{A} = (\text{End}(V))^{\otimes L}$. Hence, the definition

$$\pi(e_{\alpha_1}^{\beta_1} \otimes \dots \otimes e_{\alpha_L}^{\beta_L}) = p(\alpha_1) + p(\beta_1) + \dots + p(\alpha_L) + p(\beta_L) \tag{A.6}$$

induces a grading on \mathcal{A} regarded as a vector space. It is easy to see that an element $X = X_{\beta_1 \dots \beta_L}^{\alpha_1 \dots \alpha_L} e_{\alpha_1}^{\beta_1} \otimes \dots \otimes e_{\alpha_L}^{\beta_L} \in \mathcal{A}$ is homogeneous with parity $\pi(X)$ if and only if

$$(-1)^{\sum_{j=1}^L (p(\alpha_j) + p(\beta_j))} X_{\beta_1 \dots \beta_L}^{\alpha_1 \dots \alpha_L} = (-1)^{\pi(X)} X_{\beta_1 \dots \beta_L}^{\alpha_1 \dots \alpha_L}. \tag{A.7}$$

But the latter equation implies that two homogenous elements X and Y satisfy equation (A.2), and \mathcal{A} is a graded algebra.

The following definition of ‘graded local projection operators’ [36] provides a standard basis of the graded associative algebra \mathcal{A} which is of crucial importance in constructing solvable lattice models,

$$e_{j\alpha}^\beta = (-1)^{(p(\alpha)+p(\beta))\sum_{k=1}^{j-1} p(\gamma_k)} e_{\gamma_1}^{\gamma_1} \otimes \dots \otimes e_{\gamma_{j-1}}^{\gamma_{j-1}} \otimes e_\alpha^\beta \otimes I_{m+n}^{\otimes(L-j)}. \tag{A.8}$$

Here I_{m+n} is the $(m+n) \times (m+n)$ unit matrix, and summation over double tensor indices (i.e., over $\gamma_1, \dots, \gamma_{j-1}$) is implied. The index j on the left-hand side of (A.8) is called the site index. A simple consequence of definition (A.8) for $j \neq k$ are the commutation relations

$$e_{j\alpha}^\beta e_{k\gamma}^\delta = (-1)^{(p(\alpha)+p(\beta))(p(\gamma)+p(\delta))} e_{k\gamma}^\delta e_{j\alpha}^\beta. \tag{A.9}$$

It further follows from equation (A.8) that $e_{j\alpha}^\beta$ is homogeneous with parity

$$\pi(e_{j\alpha}^\beta) = p(\alpha) + p(\beta). \tag{A.10}$$

Hence, equation (A.9) says that odd matrices with different site indices mutually anticommute, whereas even matrices commute with each other as well as with the odd matrices. For products of matrices $e_{j\alpha}^\beta$ which are acting on the same site (A.8) implies the projection property

$$e_{j\alpha}^\beta e_{j\gamma}^\delta = \delta_\gamma^\beta e_{j\alpha}^\delta. \tag{A.11}$$

Using the super-bracket (A.3), equations (A.9) and (A.11) can be combined into

$$[e_{j\alpha}^\beta, e_{k\gamma}^\delta]_\pm = \delta_{jk} (\delta_\gamma^\beta e_{j\alpha}^\delta - (-1)^{(p(\alpha)+p(\beta))(p(\gamma)+p(\delta))} \delta_\alpha^\delta e_{j\gamma}^\beta). \tag{A.12}$$

The right-hand side of the latter equation with $j = k$ gives the structure constants of the Lie super algebra $\mathfrak{gl}(m|n)$ with respect to the basis $\{e_{j\alpha}^\beta\}$.

Since any $(m+n)$ -dimensional vector space over the complex numbers is isomorphic to \mathbb{C}^{m+n} , we may simply set $V = \mathbb{C}^{m+n}$. We may further assume that our homogeneous basis $\{e_\alpha \in \mathbb{C}^{m+n} | \alpha = 1, \dots, m+n\}$ is canonical, i.e., we may represent the vector e_α by a column vector having the only non-zero entry +1 in row α . Our basic matrices e_α^β are then $(m+n) \times (m+n)$ matrices with a single non-zero entry +1 in row α and column β .

Definition (A.8) generalizes the notion of the Jordan–Wigner transformation to systems with higher spin (see [37]). As with the Jordan–Wigner transformation another consistent definition of the graded local projection operators, also leading to (A.9) and (A.11), is obtained by placing the factors $(-1)^{(p(\alpha)+p(\beta))p(\gamma_k)} e_{\gamma_k}^{\gamma_k}$ in the tensor product on the right-hand side of (A.8) behind rather than in front of e_α^β . This alternative convention was used in [36, 37]. Here we use (A.8) instead as it naturally appears in the derivation of the algebraic Bethe ansatz for the $\mathfrak{gl}(1|2)$ generalized model with gradings $g = (-, +, -)$ and $g = (-, -, +)$ (see appendix B).

Appendix B. Derivation of the eigenvectors and eigenvalues of the transfer matrix for $g = (-, +, -)$ and $g = (-, -, +)$

It is most convenient to start with the case $g = (-, -, +)$, for which the Yang–Baxter algebra has a simple block structure, and to obtain the case $g = (-, +, -)$ afterwards. In fact, it is equivalent to the case $g = (-, -, +)$ but with a transfer matrix acting on the pseudo vacuum Ω as

$$\mathcal{T}(\lambda)\Omega = \begin{pmatrix} a_1(\lambda) & * & * \\ 0 & a_3(\lambda) & 0 \\ 0 & * & a_2(\lambda) \end{pmatrix} \Omega. \tag{B.1}$$

The ‘first level algebraic Bethe ansatz’ will be the same in both cases. The difference comes in only on the second level.

The first step of our calculation is to rewrite the graded Yang–Baxter algebra (6) with R -matrix $\check{R}_{(-\rightarrow+)}(\lambda)$ in block form: we introduce the shorthand notations

$$\begin{aligned}
 B(\lambda) &= (B_1(\lambda), B_2(\lambda)) & C(\lambda) &= \begin{pmatrix} C^1(\lambda) \\ C^2(\lambda) \end{pmatrix} \\
 D(\lambda) &= \begin{pmatrix} D_1^1(\lambda) & D_2^1(\lambda) \\ D_1^2(\lambda) & D_2^2(\lambda) \end{pmatrix}.
 \end{aligned}
 \tag{B.2}$$

Then the 3×3 monodromy matrix $T(\lambda)$ can be written as

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = \begin{pmatrix} A(\lambda) & B_1(\lambda) & B_2(\lambda) \\ C^1(\lambda) & D_1^1(\lambda) & D_2^1(\lambda) \\ C^2(\lambda) & D_1^2(\lambda) & D_2^2(\lambda) \end{pmatrix}.
 \tag{B.3}$$

The defining relations of the graded Yang–Baxter algebra (6) can be read as a 9×9 matrix equation. Let us denote the $n \times n$ unit matrix by I_n . A similarity transformation with the matrix

$$X = \begin{pmatrix} I_4 & & \\ & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \\ & & I_2 \end{pmatrix}
 \tag{B.4}$$

which cyclically permutes the 5th, 6th and 7th row and column, followed by a similarity transformations which multiplies the 5th, 8th and 9th row and column by -1 , transforms this 9×9 equation into

$$\begin{aligned}
 &\begin{pmatrix} 1 & & & & & & & & \\ & bI_2 & aI_2 & & & & & & \\ & aI_2 & bI_2 & & & & & & \\ & & & \check{r} & & & & & \end{pmatrix} \begin{pmatrix} A \otimes_{g'} \bar{A} & A \otimes_{g'} \bar{B} & B \otimes_{g'} \bar{A} & B \otimes_{g'} \bar{B} \\ A \otimes_{g'} \bar{C} & A \otimes_{g'} \bar{D} & B \otimes_{g'} \bar{C} & B \otimes_{g'} \bar{D} \\ C \otimes_{g'} \bar{A} & C \otimes_{g'} \bar{B} & D \otimes_{g'} \bar{A} & D \otimes_{g'} \bar{B} \\ C \otimes_{g'} \bar{C} & C \otimes_{g'} \bar{D} & D \otimes_{g'} \bar{C} & D \otimes_{g'} \bar{D} \end{pmatrix} \\
 &= \begin{pmatrix} \bar{A} \otimes_{g'} A & \bar{A} \otimes_{g'} B & \bar{B} \otimes_{g'} A & \bar{B} \otimes_{g'} B \\ \bar{A} \otimes_{g'} C & \bar{A} \otimes_{g'} D & \bar{B} \otimes_{g'} C & \bar{B} \otimes_{g'} D \\ \bar{C} \otimes_{g'} A & \bar{C} \otimes_{g'} B & \bar{D} \otimes_{g'} A & \bar{D} \otimes_{g'} B \\ \bar{C} \otimes_{g'} C & \bar{C} \otimes_{g'} D & \bar{D} \otimes_{g'} C & \bar{D} \otimes_{g'} D \end{pmatrix} \begin{pmatrix} 1 & & & & & & & & \\ & bI_2 & aI_2 & & & & & & \\ & aI_2 & bI_2 & & & & & & \\ & & & \check{r} & & & & & \end{pmatrix}.
 \end{aligned}
 \tag{B.5}$$

For the formula to fit on the line we suppressed the arguments and adopted the following convention: $X = X(\lambda)$, $\bar{X} = X(\mu)$ for $X = A, \dots, D$. Moreover, $a = a(\mu - \lambda)$ and $b = b(\mu - \lambda)$. The 4×4 matrix

$$\check{r} = \begin{pmatrix} 1 & & & \\ & b & a & \\ & a & b & \\ & & & b - a \end{pmatrix}
 \tag{B.6}$$

satisfies the Yang–Baxter equation. It is related to a special case (rank 2, grading $(+, -)$) of the R -matrix (1) by equation (5) and is therefore unitary,

$$\check{r}(\lambda)\check{r}(-\lambda) = I_4.
 \tag{B.7}$$

The grading $(+, -)$, corresponding to $p(1) = 0, p(2) = 1$, also appears in the graded tensor products in (B.5), where it is abbreviated as g' . These graded tensor products are defined by equation (A.4) which makes not only sense for square matrices, but for arbitrary $n \times m$ matrices. Thus, thinking of $B(\lambda)$ as a 1×2 matrix with row index 1, and similarly of $C(\lambda)$ as a 2×1 matrix with column index 1 and of $A(\lambda)$ as a 1×1 matrix with row and column index 1, all graded tensor products in (B.5) are well defined. We have, for instance,

$$B(\lambda) \otimes_{g'} C(\mu) = \begin{pmatrix} B_1(\lambda)C^1(\mu) & B_2(\lambda)C^1(\mu) \\ B_1(\lambda)C^2(\mu) & -B_2(\lambda)C^2(\mu) \end{pmatrix}. \tag{B.8}$$

We would like to remark that the defining relations of the graded Yang–Baxter algebra of the $gl(1|2)$ model, when written in block form (B.5), resemble the corresponding relations for the $gl(2)$ model.

Out of the 16 relations contained in (B.5) we shall need the following 4 for the first level algebraic Bethe ansatz:

$$B(\lambda) \otimes_{g'} B(\mu) = (B(\mu) \otimes_{g'} B(\lambda))\check{r}(\mu - \lambda) \tag{B.9}$$

$$A(\lambda) \otimes_{g'} B(\mu) = \frac{B(\mu) \otimes_{g'} A(\lambda)}{a(\lambda - \mu)} - \frac{b(\lambda - \mu)}{a(\lambda - \mu)} B(\lambda) \otimes_{g'} A(\mu) \tag{B.10}$$

$$D(\lambda) \otimes_{g'} B(\mu) = (B(\mu) \otimes_{g'} D(\lambda))\frac{\check{r}(\mu - \lambda)}{a(\mu - \lambda)} - \frac{b(\mu - \lambda)}{a(\mu - \lambda)} B(\lambda) \otimes_{g'} D(\mu) \tag{B.11}$$

$$\check{r}(\mu - \lambda)(D(\lambda) \otimes_{g'} D(\mu)) = (D(\mu) \otimes_{g'} D(\lambda))\check{r}(\mu - \lambda). \tag{B.12}$$

Note that, by (B.9), $B(\lambda)$ constitutes a representation of the Zamolodchikov algebra, and, by (B.12), $D(\lambda)$ is a representation of the Yang–Baxter algebra of the $gl(1|1)$ model.

Our goal is to calculate the eigenvectors of the transfer matrix $t(\lambda) = -A(\lambda) - \text{str}_{g'}(D(\lambda))$. In analogy with the $gl(2)$ case, we shall first of all consider the commutation relations of a multiple tensor product $B(\lambda_1) \otimes_{g'} \dots \otimes_{g'} B(\lambda_N)$ with $A(\lambda)$ and $\text{str}_{g'}(D(\lambda))$. These commutation relations can be obtained by iterating equations (B.10) and (B.11):

$$\begin{aligned} A(\lambda) \left[\bigotimes_{j=1}^N B(\lambda_j) \right] &= \left[\bigotimes_{j=1}^N B(\lambda_j) \right] A(\lambda) \prod_{j=1}^N \frac{1}{a(\lambda - \lambda_j)} \\ &\quad - \sum_{j=1}^N \left\{ B(\lambda) \otimes_{g'} \left[\bigotimes_{\substack{k=1 \\ k \neq j}}^N B(\lambda_k) \right] \right\} S_{j-1} A(\lambda_j) \frac{b(\lambda - \lambda_j)}{a(\lambda - \lambda_j)} \prod_{\substack{k=1 \\ k \neq j}}^N \frac{1}{a(\lambda_j - \lambda_k)} \end{aligned} \tag{B.13}$$

$$\begin{aligned} D(\lambda) \otimes_{g'} \left[\bigotimes_{j=1}^N B(\lambda_j) \right] &= \left\{ I_2 \otimes_{g'} \left[\bigotimes_{j=1}^N B(\lambda_j) \right] \right\} \tilde{T}(\lambda) \prod_{j=1}^N \frac{1}{a(\lambda_j - \lambda)} \\ &\quad - \sum_{j=1}^N \left\{ I_2 \otimes_{g'} B(\lambda) \otimes_{g'} \left[\bigotimes_{\substack{k=1 \\ k \neq j}}^N B(\lambda_k) \right] \right\} P_{01}(I_2 \otimes_{g'} S_{j-1}) \\ &\quad \times \{ I_2 \otimes_{g'} \text{str}_{g'}(\tilde{T}(\lambda_j)) \} \frac{b(\lambda_j - \lambda)}{a(\lambda_j - \lambda)} \prod_{\substack{k=1 \\ k \neq j}}^N \frac{1}{a(\lambda_k - \lambda_j)}. \end{aligned} \tag{B.14}$$

Here the operators $B(\lambda)$ in the multiple tensor products are multiplied in ascending order. $\tilde{T}(\lambda)$ is defined in equation (29). The operators S_{j-1} appearing on the right-hand side of (B.13) are given as

$$S_{j-1} = (\check{r}(\lambda_j - \lambda_1) \otimes_{g'} I_2^{\otimes(N-2)}) \dots (I_2^{\otimes(j-2)} \otimes_{g'} \check{r}(\lambda_j - \lambda_{j-1}) \otimes_{g'} I_2^{\otimes(N-j)}) \tag{B.15}$$

for $j = 2, \dots, N$. We further define $S_0 = \text{id}$. The use of the graded tensor product in (B.15) makes sense, since all non-zero matrix elements appear in such a way that they can be interpreted as even elements of $\text{End}(\mathcal{H})$. P_{01} is a graded transposition operator [36] on $(\text{End}(\mathbb{C}^2))^{\otimes(N+1)}$ regarded as a graded algebra (see appendix A). In the canonical basis $\{e_{\alpha_0}^{\beta_0}, \dots, e_{\alpha_N}^{\beta_N}\}$ it is expressed as

$$P_{jk} = (-1)^{p(\beta)} e_{j\alpha}^{\beta} e_{k\beta}^{\alpha} \tag{B.16}$$

with $j = 0$ and $k = 1$. In general the operators P_{jk} induce the action, of the symmetric group on the site indices of the basis elements $e_{i\alpha}^{\beta}$.

Equations (B.13) and (B.14) can be proved by induction over N (see [1]). As compared to the case $g = (+, -, -)$ treated in [1] the main difference and the main difficulty was to properly treat the graded tensor products with the induced grading g' in the derivation of equation (B.14). The remaining part of the derivation is now similar to the corresponding calculations for the case $g = (+, -, -)$. We shall discuss it only briefly.

We take the super trace in space zero of equation (B.14) and subtract it from minus one times equation (B.13). Taking into account that $t(\lambda) = -A(\lambda) - \text{str}_{g'}(D(\lambda))$ for the grading $g = (-, -, +)$ under consideration we obtain

$$\begin{aligned} t(\lambda) \left[\bigotimes_{j=1}^N g' B(\lambda_j) \right] &= \left[\bigotimes_{j=1}^N g' B(\lambda_j) \right] \\ &\times \left\{ -A(\lambda) \prod_{j=1}^N \frac{1}{a(\lambda - \lambda_j)} - \text{str}_{g'}(\tilde{T}(\lambda)) \prod_{j=1}^N \frac{1}{a(\lambda_j - \lambda)} \right\} \\ &+ \sum_{j=1}^N \left(B(\lambda) \otimes_{g'} \left[\bigotimes_{\substack{k=1 \\ k \neq j}}^N g' B(\lambda_k) \right] \right) S_{j-1} \frac{b(\lambda - \lambda_j)}{a(\lambda - \lambda_j)} \prod_{\substack{k=1 \\ k \neq j}}^N \frac{1}{a(\lambda_k - \lambda_j)} \\ &\times \left\{ A(\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^N \frac{a(\lambda_k - \lambda_j)}{a(\lambda_j - \lambda_k)} - \text{str}_{g'}(\tilde{T}(\lambda_j)) \right\}. \end{aligned} \tag{B.17}$$

This is a form rather typical of an algebraic Bethe ansatz calculation with wanted and unwanted terms on the right-hand side of the equation.

The operator $\text{str}_{g'}(\tilde{T}(\lambda))$ acts on the tensor product $(\mathbb{C}^2)^{\otimes N} \otimes \mathcal{H}$. Its eigenvectors are independent of λ since $\tilde{T}(\lambda)$ is a representation of the Yang–Baxter algebra with R -matrix $\check{r}(\lambda)$,

$$\check{r}(\mu - \lambda)(\tilde{T}(\lambda) \otimes_{g'} \tilde{T}(\mu)) = (\tilde{T}(\mu) \otimes_{g'} \tilde{T}(\lambda))\check{r}(\mu - \lambda). \tag{B.18}$$

This holds first of all for $\mathcal{T}_{g'}(\lambda)$ by construction (see (31)) and for $D(\lambda)$ by equation (B.12). It still holds after inserting the factors $(\sigma^z)^{\otimes N}$ into $D(\lambda)$ which then becomes $\mathcal{D}(\lambda)$ (see equation (30)). But, due to the factors $(\sigma^z)^{\otimes N}$ the entries of $\mathcal{D}(\lambda)$ and $\mathcal{T}_{g'}(\lambda)$ super-commute, and (B.18) holds because of (A.5).

Following Kulish and Reshetikhin [11] we define the ‘vacuum subspace’ $\mathcal{H}_0 \subset \mathcal{H}$ by the conditions

$$A(\lambda)\Phi = a_1(\lambda)\Phi \tag{B.19a}$$

$$C(\lambda)\Phi = 0 \tag{B.19b}$$

for all $\Phi \in \mathcal{H}_0$. Clearly, \mathcal{H}_0 is a linear subspace of \mathcal{H} . The following lemma [12] can be proved in a similar manner as in [1].

Lemma 1. \mathcal{H}_0 is invariant under the action of $D(\lambda)$.

Equivalently we may say that the space spanned by all linear combinations of vectors of the form $D_2^1(\mu_1) \dots D_2^1(\mu_M)\Omega$ is a linear subspace of \mathcal{H}_0 .

Suppose $\varphi\Omega \in (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{H}_0$ is an eigenvector⁴ of $\text{str}_{g'}(\tilde{T}(\lambda))$ with eigenvalue $\tilde{\Lambda}(\lambda)$. Then $\varphi\Omega$ is a 2^N -column vector with entries in \mathcal{H}_0 , and (B.19a) holds for this vector,

$$A(\lambda)\varphi\Omega = a_1(\lambda)\varphi\Omega. \tag{B.20}$$

Thus, $\varphi\Omega$ is an eigenvector of the operators in curly brackets on the right-hand side of (B.17). Since the graded tensor products of vectors $B(\lambda_j)$ form a 2^N -row vector, we conclude that

$$\left[\bigotimes_{j=1}^N {}_{g'}B(\lambda_j) \right] \varphi\Omega = \sum_{i_1, \dots, i_N=1}^2 B_{i_1}(\lambda_1) \dots B_{i_N}(\lambda_N) \varphi^{i_1, \dots, i_N} \Omega \in \mathcal{H} \tag{B.21}$$

is an eigenvector of the transfer matrix $t(\lambda)$ if the Bethe ansatz equations

$$a_1(\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^N \frac{a(\lambda_k - \lambda_j)}{a(\lambda_j - \lambda_k)} = \tilde{\Lambda}(\lambda_j) \tag{B.22}$$

are satisfied, which is just the condition for the unwanted terms in the second curly bracket on the right-hand side of (B.17) to vanish. The corresponding eigenvalue of $t(\lambda)$ is

$$\Lambda(\lambda) = -a_1(\lambda) \prod_{j=1}^N \frac{1}{a(\lambda - \lambda_j)} - \tilde{\Lambda}(\lambda) \prod_{j=1}^N \frac{1}{a(\lambda_j - \lambda)}. \tag{B.23}$$

The remaining task is to solve the eigenvalue problem of $\text{str}_{g'}(\tilde{T}(\lambda))$ on the space $(\mathbb{C}^2)^{\otimes N} \otimes \mathcal{H}_0$. This task can be accomplished by a second Bethe ansatz which is possible, because $\tilde{T}(\lambda)$ is a representation of the Yang–Baxter algebra (see (B.18)) and the vector

$$\hat{\Omega} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes N} \Omega \tag{B.24}$$

is a highest vector for $\tilde{T}(\lambda)$. In fact, introducing the explicit form of the L -matrix (33) and the explicit form of $\mathcal{D}_{(-\rightarrow)}$, equation (30c), into definition (29) of $\tilde{T}(\lambda)$ we obtain

$$\tilde{T}(\lambda)\hat{\Omega} = \begin{pmatrix} a_2(\lambda) & * \\ 0 & a_3(\lambda) \prod_{j=1}^N a(\lambda_j - \lambda) \end{pmatrix} \hat{\Omega}. \tag{B.25}$$

For the construction of the eigenvectors of $\text{str}_{g'}(\tilde{T}(\lambda))$ we extract the following commutation relations from (B.18):

⁴ We loosely write $\varphi\Omega$ instead of $\varphi \otimes \Omega$.

$$\tilde{B}(\lambda)\tilde{B}(\mu) = \tilde{B}(\mu)\tilde{B}(\lambda) (b(\mu - \lambda) - a(\mu - \lambda)) \tag{B.26a}$$

$$(\tilde{A}(\lambda) - \tilde{D}(\lambda))\tilde{B}(\mu) = \frac{\tilde{B}(\mu)(\tilde{A}(\lambda) - \tilde{D}(\lambda))}{a(\lambda - \mu)} - \frac{b(\lambda - \mu)}{a(\lambda - \mu)}\tilde{B}(\lambda)(\tilde{A}(\mu) - \tilde{D}(\mu)). \tag{B.26b}$$

Here we referred back to the notation for the matrix elements introduced in (29). Iterating (B.26b) we obtain

$$\begin{aligned} \text{str}_{g'}(\tilde{T}(\lambda)) \left[\prod_{k=1}^M \tilde{B}(\mu_k) \right] &= \left[\prod_{k=1}^M \tilde{B}(\mu_k) \right] \text{str}_{g'}(\tilde{T}(\lambda)) \prod_{k=1}^M \frac{1}{a(\lambda - \mu_k)} \\ &\quad - \sum_{k=1}^M \left[\tilde{B}(\lambda) \prod_{\substack{l=1 \\ l \neq k}}^M \tilde{B}(\mu_l) \right] s_{k-1} \text{str}_{g'}(\tilde{T}(\mu_k)) \frac{b(\lambda - \mu_k)}{a(\lambda - \mu_k)} \prod_{\substack{l=1 \\ l \neq k}}^M \frac{1}{a(\mu_k - \mu_l)} \end{aligned} \tag{B.27}$$

where the products over the $\tilde{B}(\mu_k)$ are ordered in ascending order and by definition

$$s_{k-1} = \prod_{l=1}^{k-1} (b(\mu_k - \mu_l) - a(\mu_k - \mu_l)) \tag{B.28}$$

for $k = 2, \dots, M$ and $s_0 = 1$. It follows that

$$\varphi\Omega = \tilde{B}(\mu_1) \dots \tilde{B}(\mu_M)\hat{\Omega} \in (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{H}_0 \tag{B.29}$$

is an eigenvector of $\text{str}_{g'}(\tilde{T}(\lambda))$ with eigenvalue

$$\tilde{\Lambda}(\lambda) = \left[a_2(\lambda) - a_3(\lambda) \prod_{j=1}^N a(\lambda_j - \lambda) \right] \prod_{k=1}^M \frac{1}{a(\lambda - \mu_k)} \tag{B.30}$$

if the Bethe ansatz equations

$$\frac{a_3(\mu_k)}{a_2(\mu_k)} = \prod_{j=1}^N \frac{1}{a(\lambda_j - \mu_k)} \quad k = 1, \dots, M \tag{B.31}$$

are satisfied. Inserting expressions (B.29) into (B.21) and (B.30) into (B.22), (B.23) we arrive at the results shown in section 4 and our derivation for the case $g = (-, -, +)$ is complete.

It is now relatively easy to perform the algebraic Bethe ansatz for the remaining grading $(-, +, -)$. According to our remark at the beginning of this appendix this case is equivalent to considering the grading $(-, -, +)$ with the vacuum action shown in equation (B.1). Therefore only the second level Bethe ansatz, starting from equation (B.24), has to be modified. Since $D(\lambda)$ now acts as a lower triangular matrix on Ω we must choose the auxiliary vacuum for the second Bethe ansatz as $\hat{\Omega} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes N} \Omega$. Then $\tilde{T}(\lambda)$ has lower triangular action on $\hat{\Omega}$, and an algebraic Bethe ansatz (with \tilde{C} replacing \tilde{B}) becomes again possible: introducing the explicit form of the L -matrix (33) and the explicit form of $\mathcal{D}_{(-+-)}$, equation (30b), into definition (29) of $\tilde{T}(\lambda)$ we obtain

$$\tilde{T}(\lambda)\hat{\Omega} = \begin{pmatrix} a_3(\lambda) \prod_{j=1}^N a(\lambda_j - \lambda) & 0 \\ * & a_2(\lambda) \prod_{j=1}^N \frac{a(\lambda_j - \lambda)}{a(\lambda - \lambda_j)} \end{pmatrix} \hat{\Omega} \tag{B.32}$$

where we used the identity $a(\lambda) - b(\lambda) = a(\lambda)/a(-\lambda)$. Instead of the commutation relations (B.26) we now need the commutation relation between $\tilde{C}(\lambda)$ and $\tilde{C}(\mu)$ and between $\text{str}_{g'}(\tilde{T}(\lambda))$ and $\tilde{C}(\mu)$. These commutation relations are again contained in (B.18). They are of the same

form as (B.26) with \tilde{C} replacing \tilde{B} and the arguments λ , μ of the functions a and b interchanged. It follows that

$$\varphi\Omega = \tilde{C}(\mu_1) \dots \tilde{C}(\mu_M)\hat{\Omega} \in (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{H}_0 \quad (\text{B.33})$$

is an eigenvector of $\text{str}_g(\tilde{T}(\lambda))$ with eigenvalue

$$\tilde{\Lambda}(\lambda) = \left[a_3(\lambda) \prod_{j=1}^N a(\lambda_j - \lambda) - a_2(\lambda) \prod_{j=1}^N \frac{a(\lambda_j - \lambda)}{a(\lambda - \lambda_j)} \right] \prod_{k=1}^M \frac{1}{a(\mu_k - \lambda)} \quad (\text{B.34})$$

if the Bethe ansatz equations

$$\frac{a_3(\mu_k)}{a_2(\mu_k)} = \prod_{j=1}^N \frac{1}{a(\mu_k - \lambda_j)} \quad k = 1, \dots, M \quad (\text{B.35})$$

are satisfied. Inserting expressions (B.33) into (B.21) and (B.34) into (B.22) and (B.23) we obtain the results for $g = (-, +, -)$ shown in section 4.

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